Inequality, Poverty, Two Invariance Conditions, and a Product Rule

By/Par **S. Subramanian**
Independent Researcher,
Chennai, Tamil Nadu, India
ssubramanianecon@gmail.com , subbu@mids.ac.in

ABSTRACT

Two axioms in the measurement of inequality and poverty which are widely perceived to be innocuous and unexceptionable - although they have both been challenged in the literature - are the Scale Invariance Axiom and the Replication Invariance Axiom. These axioms have endorsed an essentially *relative* approach (with respect to income-size and population-size respectively) to the measurement of inequality and poverty. The present paper is an expository essay which aims to clarify the logical and ethical limitations of either a purely relative or a purely absolute approach to distributional measurement. In the process, it also reviews two proposals - due to Manfred Krtscha and Eduardo Arriaga respectively - for 'intermediate' measures of inequality and poverty, which moderate the 'extreme' values underlying relative and absolute measures by combining these opposing values in a simple product formula.

Key Words: Scale Invariance, Translation Invariance, Unit Consistency, Replication Invariance, Replication Scaling, Population Replication Principle

RESUME

Les axiomes d'échelle invariante et de réplication invariante, sont deux axiomes de la mesure de l'inégalité et de la pauvreté largement perçus comme inoffensifs et irrécusables, bien qu'ils aient tous deux été contestés dans la littérature. Ces axiomes ont appuyé une approche essentiellement relative (par rapport au revenu et à la population, respectivement) de l'inégalité et de la pauvreté. Cet article vise à clarifier les limites logiques et éthiques d'une mesure soit purement relative, soit purement absolue. Il examine également deux propositions - suivant Manfred Krtscha et Eduardo Arriaga respectivement - des mesures «intermédiaires» de l'inégalité et de la pauvreté, qui modèrent les valeurs «extrêmes» des mesures sous-jacentes relatives et absolues en combinant ces valeurs opposées dans une formule simple.

Mots-clés: Échelle d'Invariance, Invariance, Unité de cohérence, Réplication Invariante, Echelle de réplication, Principe de réplication de la population.

JEL Classification: B40, D30, D31, D63, I32, O15.

Acknowledgement: The author thanks Debabrata Pal for helpful comments on an earlier version of the paper.

Policeman: What are you doing with that elephant?
Character played by Jimmy Durante: Elephant? What elephant?
- Billy Rose's Jumbo

1. INTRODUCTION

Two conceptual problems - among several others - which confront the analyst of inequality and poverty are what may be called the Problem of Variable Size and the Problem of Variable Populations. Stated loosely, the Variable Size Problem is concerned with how our judgements on inequality and poverty ought to respond to changes in the size of the *distribution* under consideration, while the Variable Populations Problem is concerned with how our judgements on inequality and poverty ought to respond to changes in the size of the *population* under consideration. In this paper, we shall be concerned with the Variable Size Problem as it applies to the measurement of inequality, and the Variable Populations Problem as it applies to the measurement of poverty.

In the literature on the measurement of inequality and poverty, the approach most widely resorted to in addressing the two Problems mentioned above has been to invoke two specific invariance postulates - the Scale Invariance Axiom and the Replication Invariance Axiom respectively. In the context of inequality measurement, Scale Invariance requires that the value of the chosen measure should be invariant with respect to any uniform scaling up or scaling down of a distribution. In the context of poverty measurement, Replication Invariance requires that the value of the chosen measure should be invariant with respect to any uniform replication of the population. The two invariance conditions have ensured that the favoured interpretation of inequality and poverty indices, in the bulk of the relevant literature, has been in terms of these being *relative* rather than *absolute* indices.

The predominant view which prevails in the literature has not, however, gone unchallenged. Where inequality is concerned, there is a long - even if not widely accepted nor even acknowledged - tradition that has dealt with the merits of absolute, as also of 'intermediate', measures of inequality. Salient contributions to this literature would include the work of, among others, Kolm (1976a, 1976b), Moyes (1987), Bossert (1990a), Bossert and Pfingsten (1990), Krtscha (1994), Chakravarty and Tyagarupananda (1998, 2009), Del Rio and Ruiz-Castillo (2000, 2001), Atkinson and Brandolini (2004), Yoshida (2005), Jenkins and Jantti (2005), Zheng (2007), Del Rio and Alonso-Villar (2008, 2011), Azpitarte and Alonso-Villar (2011), Bosmans, Decancq and Decoster (2011), and, importantly, Zoli (2012). Similarly, where poverty (and a set of related issues) is concerned, there is a strand of the literature which has questioned an unqualified endorsement of the Replication Invariance Axiom. The relative approach to poverty assessment, with its emphasis on the *proportion* of the population in poverty, has been sought to be contrasted with an absolute approach that emphasizes the *numbers* of people in poverty, or some less extreme 'intermediate' compromise between the relative and the absolute approaches. Contributions to this strand of the literature would include, among others, Kundu and Smith (1983), Bossert (1990b), Paxton (2003), Chakravarty, Kanbur and Mukherjee (2006), Kanbur and Mukherjee (2007), Hassoun (2010), Hassoun and Subramanian (2012), Subramanian (2000, 2002, 2005a, 2005b, 2006, 2012, 2013a, 2013b), and Zoli (2009).

It is worth noting that a distinguished response to the Variable Size Problem in inequality measurement is constituted by an 'intermediate' measure due to Krtscha (1994), which is simply the product of two other very well-known measures of inequality, one of which - the coefficient of variation - is a relative measure, and the other - the standard deviation - is an absolute measure. Similarly, an appealing 'resolution' of the Variable Populations Problem in poverty measurement (originally conceived of in the context of a precisely analogous problem in the measurement of *urbanization*) is due to Arriaga (1970), who pointed to the plausibility of a measure which happens to be, quite simply, a product of two very well-known measures, one of which - the headcount ratio - is a relative measure, and the other - the aggregate headcount - is an absolute measure.

The foregoing introduction is an account of the principal ingredients of this paper. These are:

(a) two widely-employed invariance conditions in the axiomatics of inequality and poverty measurement, namely the Scale Invariance and the Replication Invariance conditions, which are compatible with the advancement of *relative* measures;

(b) the postulation of alternative Variable Size and Variable Populations axioms which are compatible with the advancement of *absolute* measures; and

(c) the further postulation of *intermediate* measures, such as the Krtscha Index of Inequality and the Arriaga Index of Urbanization/Poverty which are essentially manifestations of a simple product rule in terms of which the relevant intermediate measures are derived as a product of the corresponding relative and absolute measures.

The paper is organized as follows. The introductory section is followed by one dealing with the preliminaries of concepts and definitions. Next, the Variable Size Problem in inequality measurement is reviewed, and this is succeeded by a treatment of the Variable Populations Problem in poverty measurement. A final section discusses the analogous nature of the two Problems and their similar resolution through the deployment of a 'product rule', whereby relative and absolute approaches to assessment are sought to be reconciled in some appropriate 'intermediate' compromise.

This paper is written on the premise that it is a matter of some importance to assimilate the employment of intermediate measures of inequality and poverty on a routine basis into mainstream theoretical and empirical studies of the phenomena of inequality and poverty, as constituting sensibly moderate compromises between the arguably 'extreme' value-orientation of purely relative and purely absolute measures. This will entail a quick review of the previous work of a number of writers on the subject of enquiry. The objective of the paper will therefore be geared to the ends of recollection and exposition of an important though arguably relatively neglected strand of the literature.

2. PRELIMINARY FORMALITIES

We let x_i stand for the (non-negative) income of person i $(i = 1,...,n)$ in a community of n (≥ 1) individuals constituting a set designated by N. An income distribution is an n-vector of incomes $\mathbf{x} = (x_1,..., x_i,..., x_n)$. $\mathbf{X}_n$ is the set of all income n-vectors, and $\mathbf{X}$ is the set of all conceivable income distributions, given by: $\mathbf{X} = \bigcup_{n \in \mathrm{N}} \mathbf{X}_n$, where N is the set of positive integers. We shall let R stand for the set of real numbers, and R_{++} for the set of positive reals. A *poverty line* is a positive real number z such that individuals with incomes less than z are certified to be *poor*. Given any set of individuals *N*, the set of poor individuals in *N* is designated by *Q*(*N*), and the set of non-poor individuals by $R(N)$ (so $Q(N) \bigcup R(N) \equiv N)$. For all $\mathbf{x} \in \mathbf{X}$, $\mathbf{x}_Q$ will stand for the sub-vector of poor incomes, and $\mathbf{x}_R$ for the sub-vector of non-poor incomes (so $\mathbf{x} \equiv (\mathbf{x}_Q, \mathbf{x}_R)$). For all $\mathbf{x} \in \mathbf{X}$, $\mu(\mathbf{x})$ will stand for the mean, and $n(\mathbf{x})$ for the dimensionality, of the distribution **x**. Also, a distribution **x** is a *perfectly equal distribution* if $x_i = \mu(\mathbf{x})$ for all $i = 1,...,n(\mathbf{x})$; and it is a *perfectly concentrated distribution* if $[x_i = 0$ for all $i = 1,...,n(\mathbf{x}) \setminus \{j\}$ and $x_j = n(\mathbf{x})\mu(\mathbf{x})$ for some *j*].

An *inequality measure* is a mapping $I : \mathbf{X} \to \mathrm{R}$ such that, for every income-vector **x** belonging to **X**, *I* specifies a unique real number which is intended to capture the extent of inequality associated with **x**. A *poverty measure* is a mapping $P : \mathbf{X} \times \mathrm{R}_{++} \to \mathrm{R}$ such that, for every income-vector **x** belonging to **X** and every (positive) poverty line *z*, *P* specifies a unique real number which is intended to capture the extent of poverty associated with the regime $(\mathbf{x}; z)$.

An inequality measure will be said to satisfy (a) the *Transfer Axiom* (*Axiom T*) if its value declines in the presence, other things equal, of a rank-preserving progressive (that is, richer person-to-poorer person) transfer of income; (b) the *Symmetry Axiom* (*Axiom S*) if its value is invariant with respect to a permutation of incomes among individuals; (c) the *Continuity Axiom* (*Axiom C*) if it is continuous on $\mathbf{X}_n$ for every $n \in \mathrm{N}$ (so that minor changes in an income distribution do not produce discontinuously abrupt changes in the value of the measure); and (d) the *Positive Responsiveness Axiom* (*Axiom PR*) if, in a two-person world, its value increases when, other things equal, the richer person's income increases. We shall confine attention to the class of inequality measures which are symmetric. Arguably, Transfer, Symmetry, Continuity and Positive Responsiveness are reasonable properties for an inequality index to satisfy.

Given any poverty line *z*, a poverty measure will be said to satisfy (a) the *Income Focus Axiom* (*Axiom IF*) if its value is invariant with respect to an increase in a non-poor person's income; (b) the *Symmetry Axiom* (*Axiom S**), if its value is invariant with respect to a permutation of incomes among individuals; (c) the *Continuity Axiom* (*Axiom C**) if it is continuous in poor incomes; (d) the *Monotonicity Axiom* (*Axiom M*) if its value declines, other things equal, with an increase in the income of a poor person; and (e) The *Weak Transfer Axiom* (*Axiom T**) if its value declines when, other things equal, there is a transfer of income from a non-poor person to a poor person that keeps the numbers of individuals on either side of the poverty line unchanged. We shall confine attention to the class of poverty measures which are symmetric. Arguably, Symmetry, Monotonicity, Continuity and Weak Transfer are reasonable properties for a poverty index to satisfy.

3. INEQUALITY AND THE VARIABLE SIZE PROBLEM

How (if at all) should the value of an inequality measure change with a change in the size of a distribution? The most commonly employed axiom, in this context, is the so-called *Scale Invariance Axiom* (*Axiom SI*):

Scale Invariance (*Axiom SI*): An inequality measure $I : \mathbf{X} \to \mathrm{R}$ satisfies Scale Invariance if and only if, for all $\mathbf{x} \in \mathbf{X}$ and all $\lambda \in \mathrm{R}_{++}$, $I(\mathbf{x}) = I(\lambda\mathbf{x})$.

Under Scale Invariance, inequality in the distribution of a cake is independent of its size so long as the proportions of cake going to each individual are preserved: equal proportionate increases in the amount going to each individual will leave the extent of measured inequality unchanged. Measures of inequality which satisfy Scale Invariance are also called *relative* measures. A well-known relative measure of inequality is the *coefficient of variation* (*CV*), which is given, for all $\mathbf{x} \in \mathbf{X}$, by:

$$(3.1)\quad CV(\mathbf{x}) = \left[\frac{1}{n(\mathbf{x})} \sum_{i=1}^{n(\mathbf{x})} \left\{ \frac{\mu(\mathbf{x}) - x_i}{\mu(\mathbf{x})} \right\}^2 \right]^{1/2}.$$

Scale Invariance also implies *neutrality to the units of measurement*, namely the property that the value of an inequality measure should be invariant with respect to the units (for example, rupees or francs) in which income is measured. Such neutrality has often been considered to be an indispensable feature of an inequality index. It is possible that this belief has played a major role in explaining the near-universal appeal which relative measures of inequality seem to enjoy in the inequality literature.

In this connection, it must be pointed out that neutrality to the units of measurement might be a needlessly strong requirement. Zheng (2007) has advanced an attractively weaker condition called *Unit Consistency*, which requires not the cardinal property of *value*-neutrality but the ordinal property of *ranking*-neutrality: that is to say, unit-consistency demands only that the ranking (and not necessarily the value) of any pair of distributions should be invariant with respect to the units in which income is measured.

Unit Consistency (Axiom UC). An inequality measure $I : \mathbf{X} \to \mathrm{R}$ satisfies Unit Consistency if and only if, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, if $I(\mathbf{x}) \geq I(\mathbf{y})$, then $I(\lambda\mathbf{x}) \geq I(\lambda\mathbf{y})$ for any $\lambda \in \mathrm{R}_{++}$.

The perceived virtue of the property of value-neutrality has arguably precluded sufficient engagement with the positive and normative limitations of relative inequality measures. At the positive level, and as Subramanian (2013a) points out in the context of two-person distributions, Scale Invariance is incompatible with Positive Responsiveness. (If $\mathbf{x} = (0,1)$ and $\mathbf{y} = (0,100)$, then $I(\mathbf{x}) = I(\mathbf{y})$ by Scale Invariance, whereas $I(\mathbf{x}) < I(\mathbf{y})$ by Positive Responsiveness.) If one holds the view that it is hard to quarrel with the reasonableness of Positive Responsiveness, then this does leave Scale Invariance somewhat exposed. At the normative level, and as Kolm (1976a,b) has pointed out, in the presence of income-growth, relative measures of inequality are pronouncedly ‘rightist’ in their value-orientation: a relative measure cannot differentiate between the distributions $\mathbf{x} = (1,50)$ and $\mathbf{y} = (2,100)$, even though the absolute gap between the two persons' incomes is twice as large in the distribution **y** as in the distribution **x**. Perhaps as a reaction to the extreme conservatism (under income-growth) of Scale Invariance, an alternative invariance condition has been postulated in the literature, one which sees invariance as being preserved not under equal *proportionate* increases in all incomes, but under equal *absolute* increases in all incomes. This is the property of *Translation Invariance* (see Kolm 1976a, b):

Translation Invariance (*Axiom TI*): An inequality measure $I : \mathbf{X} \to \mathrm{R}$ satisfies Translation Invariance if and only if, for all $\mathbf{x} \in \mathbf{X}$ and all $t \in \mathrm{R}$, $I(\mathbf{x}) = I(\mathbf{x} + \mathbf{t})$, where $\mathbf{t} \equiv (t,...,t)$ and $n(\mathbf{t}) = n(\mathbf{x})$.

Under Translation Invariance equal absolute increases or decreases in the amount of income going to each individual will leave the extent of measured inequality unchanged. Measures of inequality which satisfy Translation Invariance are also called *absolute* measures. A well-known absolute measure of inequality is the *standard deviation of incomes* (SD), a mean-dependent measure which is simply the coefficient of variation scaled by the mean income and given, for all $\mathbf{x} \in \mathbf{X}$, by:

$$(3.2)\quad SD(\mathbf{x}) = \left[\frac{1}{n(\mathbf{x})} \sum_{i=1}^{n(\mathbf{x})} \{\mu(\mathbf{x}) - x_i\}^2 \right]^{1/2}.$$

Absolute measures of inequality, being mean-dependent, will obviously violate the property of neutrality to units of measurement, although they need not fall foul of the property of Unit Consistency: this, as it happens, is true for the Standard Deviation measure. Thus, if Unit Consistency is seen as an adequate substitute for the arguably excessively strong condition of value-neutrality, then an absolute measure of inequality such as the standard deviation might be seen as an acceptable alternative to the more standard formulation of relative measures. However, it can be shown that absolute measures, like relative measures, are also suspect from both positive and normative perspectives.

Subramanian (2013a) has shown, by means of a simple example relating to two-person distributions, that a Translation-Invariant inequality measure could violate the property of Continuity if it is also required to be able to differentiate a perfectly equal distribution of incomes from a perfectly concentrated one. (If $\mathbf{x} = (0,1), \mathbf{v} = (99,100)$ and $\mathbf{y} = (100,100)$, then $I(\mathbf{x}) = I(\mathbf{v})$ by Translation Invariance, and $I(\mathbf{v}) \approx I(\mathbf{y})$ by Continuity, whence $I(\mathbf{x}) \approx I(\mathbf{y})$, that is, a perfectly concentrated distribution (**x**) is assessed as displaying *virtually* the same extent of inequality as a perfectly equal distribution (**y**).) Apart from this problem of logical coherence, absolute measures may also fail to be wholly normatively appealing. It is well known that under Translation Invariance, equal decrements in income must be treated the same way as equal increments, so that the distribution **x** = (1 million, 2 million) must be judged to exhibit the same extent of inequality as the distribution **y** = (0, 1 million): this surely involves the assertion of a questionable moral equivalence, given that **x** is a distribution describing two millionaires while **y** is a distribution relating to a destitute and a millionaire.

If Kolm pronounced relative inequality measures to be 'rightist' in the presence of income-growth, he also pronounced absolute inequality measures to be 'leftist' in the presence of income-growth (Kolm 1976a, b). This leads to the question: is there a moderate middle-ground between the extreme value-orientations of rightist measures and leftist measures? Such a compromise might be located in what Kolm (1976a, b) refers to as '*intermediate inequality measures*', which are measures that register a rise in value with an equal proportionate increase in all incomes, and a decline in value with an equal absolute increase in all incomes:

Intermediate Inequality Measure: An inequality measure $I:\mathbf{X}\to\mathrm{R}$ is intermediate if and only if, for all distributions $\mathbf{x}\in\mathbf{X}$ which are not perfectly equal distributions and all $\lambda, t\in\mathrm{R}_{++}$, (a) $I(\mathbf{x})<I(\lambda\mathbf{x})$ and (b) $I(\mathbf{x})>I(\mathbf{x}+\mathbf{t})$, where $\mathbf{t}\equiv(t,...,t)$ and $n(\mathbf{t})=n(\mathbf{x})$.

Krtscha (1994) has proposed an approach to the Variable Size Problem which entails an invariance postulate of the following nature (explicated in Subramanian 2013b). Suppose **x** and **y** are two distributions of the same dimensionality *n* and with means $\mu(\mathbf{x})$ and $\mu(\mathbf{y})$ respectively, with $\mu(\mathbf{y})>\mu(\mathbf{x})$. Let $\mathbf{x}'$ be a distribution obtained from **x** in such a way that the first incremental rupee in the transition from **x** to **y** is split into two halves, with the first half distributed in the same proportions as in **x** and the second half divided equally among the population. Let $\mathbf{x}''$ be a distribution obtained from $\mathbf{x}'$ in such a way that the second incremental rupee in the transition from **x** to **y** is split into two halves, with the first half distributed in the same proportions as in $\mathbf{x}'$ and the second half divided equally among the population. Let $\mathbf{x}'''$ be similarly derived from $\mathbf{x}''$, …, and so on, incremental rupee after incremental rupee, until we arrive at the last marginal rupee and the corresponding distribution $\mathbf{y}'$ (where, of course, $\mu(\mathbf{y}')=\mu(\mathbf{y})$). Suppose we now require that $I(\mathbf{x})=I(\mathbf{x}')=I(\mathbf{x}'')=I(\mathbf{x}''')=...=I(\mathbf{y}')$. Then, effectively, we are postulating an 'intermediate' or 'centrist' invariance condition which strikes a middle ground between the requirements of Scale Invariance and Translation Invariance. Call this property 'Centrist Invariance', which is a midway requirement between the 'rightist' and 'leftist' orientations of relative and absolute measures respectively. Krtscha (1994) demonstrates that there exists an inequality measure - let us call this the Krtscha Measure (*K*) - satisfying Centrist Invariance and a number of other attractive properties, including in particular the property of Unit Consistency, which is given, for all $\mathbf{x}\in\mathbf{X}$, by

(3.3) $$K(\mathbf{x})=\frac{1}{n(\mathbf{x})\mu(\mathbf{x})}\sum_{i=1}^{n(\mathbf{x})}\{\mu(\mathbf{x})-x_i\}^2 .$$

From (3.1), (3.2) and (3.3), it is very easy to see that, for all $\mathbf{x}\in\mathbf{X}$:

(3.4) $K(\mathbf{x})=CV(\mathbf{x})\cdot SD(\mathbf{x})$:

The (intermediate) Krtscha index is just the product of two very well-known inequality measures: the (relative) Coefficient of Variation and the (absolute) Standard Deviation. Indeed, as noted in Subramanian (2013b), the product of any relative measure and any absolute measure must yield an intermediate measure: this is because an equi-proportionate increase of all incomes will leave the value of the relative measure unchanged while raising the value of the absolute measure and so raising the value of the product of the two, while an equal absolute increase of all incomes will leave the value of the absolute measure unchanged while lowering the value of the relative measure and so lowering the value of the product of the two - which is the distinguishing property of intermediate measures of inequality.

Briefly, it turns out that a simple product rule is at the heart of an appealing resolution of the Variable Size Problem, as reflected in the (sadly under-utilized) Krtscha measure of inequality.

4. POVERTY AND THE VARIABLE POPULATIONS PROBLEM

How should poverty comparisons be carried out when they are between variable, rather than fixed, populations? The most widely-favoured approach to bridging the transition from fixed to variable population poverty comparisons is *via* the so-called Replication Invariance Axiom which requires measured poverty to be invariant with respect to replications of income distributions:

Replication Invariance (*Axiom RI*): A poverty measure $P:\mathbf{X}\times \mathrm{R}_{++}\to \mathrm{R}$ satisfies Replication Invariance if and only if, for all $\mathbf{x},\mathbf{y}\in \mathbf{X}$ and $\mathrm{z}\in \mathrm{R}_{++}$, if $\mathbf{y}=(\mathbf{x},...,\mathbf{x})$ and $n(\mathbf{y})=kn(\mathbf{x})$ for any $k\in \mathrm{N}$, then $P(\mathbf{x};\mathrm{z})=P(\mathbf{y};\mathrm{z})$.

Replication Invariance, in other words, requires us to take a per capita view of poverty. Indeed, it is at the heart of comparisons of distributions in terms of such criteria as Stochastic and Lorenz Dominance. The most elementary measure of poverty one can think of is some headcount of those in poverty. Under Replication Invariance, one would be obliged to take a *relative* view of the headcount, namely a view which focuses on the *proportion* of the population in poverty. Such a view is captured in the very widely employed poverty measure called the *headcount ratio* (P_H) which is given, for all $(\mathbf{x};z)\in \mathbf{X}\times \mathrm{R}_{++}$, by:

(4.1) $P_H(\mathbf{x};z)=n(\mathbf{x}_{\mathrm{Q}};z)/n(\mathbf{x};z)$.

A widely-endorsed property of poverty indices is the so-called Income Focus Axiom which was referred to in the introductory section of this paper. Income Focus requires a poverty index to be invariant with respect to increases in non-poor incomes:

Income Focus (*Axiom IF*): A poverty measure $P:\mathbf{X}\times \mathrm{R}_{++}\to \mathrm{R}$ satisfies Income Focus if and only if, for all $\mathbf{x},\mathbf{y}\in \mathbf{X}$ and $\mathrm{z}\in \mathrm{R}_{++}$, if [$n(\mathbf{y})=n(\mathbf{x})$ and $y_i=x_i \forall i\in N(\mathbf{x})\setminus\{j\}$ & $y_j>x_j$ for some *j* satisfying $x_j>z$], then $P(\mathbf{x};z)=P(\mathbf{y};z)$.

The rationale for this property resides in Sen's (1981) view that poverty is a feature of the *poor* population, and that therefore assessments of poverty ought to be focused on *poor* incomes only. By the same logic, however, one should also endorse a *Population Focus Axiom* - see Hassoun and Subramanian (2012) and Subramanian (2012) - namely the view that increases in the non-poor *population* ought to affect measured poverty no more than increases in non-poor *incomes*:

Population Focus (*Axiom PF*): A poverty measure $P:\mathbf{X}\times \mathrm{R}_{++} \rightarrow \mathrm{R}$ satisfies Population Focus if and only if, for all $\mathbf{x},\mathbf{y} \in \mathbf{X}$ and $z \in \mathrm{R}_{++}$, if [$\mathbf{y} = (\mathbf{x}, x)$ for any $x > z$], then $P(\mathbf{x};z) = P(\mathbf{y};z)$.

A headcount measure of poverty which satisfies the Population Focus axiom is the *absolute number*, rather than *proportion*, of the poor population: this is the so-called *aggregate headcount* measure (P_A) which is given, for all $(\mathbf{x};z) \in \mathbf{X}\times \mathrm{R}_{++}$, by:

(4.2) $P_A(\mathbf{x};z) = n(\mathbf{x}_{\mathrm{Q}};z)$.

We have noted, in the context of inequality measurement and the Variable Size Problem, the sorts of logical difficulties that can arise when we seek to combine a principle compatible with Scale Invariance with a principle compatible with Translation Invariance. Analogously, and in the context of the Variable Populations Problem currently under discussion, one encounters a tension between the relative view of poverty afforded by Replication Invariance and the absolute view of poverty afforded by Population Focus. In particular, and as shown in Hassoun and Subramanian (2012), there exists no poverty measure satisfying the axioms of Replication Invariance, Monotonicity and Population Focus. (If z = 2, $\mathbf{x}$ = (1,1), $\mathbf{y}$ = (1,3), and $\mathbf{v}$ = (1), then $P(\mathbf{x};z) > P(\mathbf{y};z)$ by Monotonicity; $P(\mathbf{x};z) = P(\mathbf{v};z)$ by Replication Invariance; and therefore $P(\mathbf{v};z) > P(\mathbf{y};z)$ - which, however, contradicts $P(\mathbf{v};z) = P(\mathbf{y};z)$, as dictated by Population Focus.) A corollary to this proposition is that there exists no poverty measure which satisfies the axioms of Replication Invariance, Weak Transfer and Population Focus. (This follows from the fact - see Hassoun and Subramanian (2012) for a simple demonstration - that Replication Invariance and Weak Transfer together imply Monotonicity; and we already know that Monotonicity, Replication Invariance and Population Focus are mutually incompatible.) Monotonicity and Transfer were seen by Sen (1976) as crucially desirable properties of a poverty measure, a view that has seldom been challenged in the poverty measurement literature. The fact that neither Monotonicity nor Weak Transfer is compatible with the combination of Replication Invariance and Population Focus suggests that the latter two invariance conditions are less than wholly appealing.

Furthermore, one can also call into question the normative appeal of Replication Invariance and Population Focus. In particular, Broome (1996) proposed a principle in population ethics which he called the *Constituency Principle*. This is the requirement that in comparing the 'goodness' of alternative states of the world we should only focus attention on the goodness of these states for some appropriately identified constituency of individuals which alone is regarded as relevant for the comparison. This is a version of a Focus Axiom in poverty measurement, which suggests that the only constituency of relevance for a comparison of the extent of poverty in two states of the world is the constituency of the *poor* in the two states. An alternative normative principle one may advance is that the extent of poverty in any state of the world should be seen to be an increasing function of the probability of encountering a poor person in that state. This is what one may call a *Likelihood Principle* (see Subramanian 2002, 2005a and 2005b). Unfortunately, and as a little reflection will confirm, an invariance condition such as Replication Invariance violates the Constituency Principle (though it satisfies the Likelihood Principle), while an invariance condition such as Population Focus violates the Likelihood Principle (though it satisfies the Constituency Principle).

Notice further that a variable populations principle which is compatible with the Population Focus Axiom, and which would also be satisfied by the aggregate headcount measure, is one which would require a *k*-fold replication of an income distribution to be accompanied by a *k*-fold increase in the value of the poverty measure. This requirement is captured in the *Replication Scaling Axiom* (Subramanian 2002):

Replication Scaling (*Axiom RS*): A poverty measure $P: \mathbf{X} \times \mathrm{R}_{++} \rightarrow \mathrm{R}$ satisfies Replication Scaling if and only if, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and $\mathrm{z} \in \mathrm{R}_{++}$, if $\mathbf{y} = (\mathbf{x}, ..., \mathbf{x})$ and $n(\mathbf{y}) = kn(\mathbf{x})$ for any $k \in \mathrm{N}$, then $P(\mathbf{x}; \mathrm{z}) = kP(\mathbf{y}; \mathrm{z})$.

In view of the problems of both logical and normative appeal associated with Replication Invariance, at one extreme, and Replication Scaling, at the other extreme, there may be a case for considering a population principle which is intermediate in orientation between these two invariance conditions. Such a principle is the axiom of *Felixible Replication Responsiveness*, advanced in Subramanian (2005a):

Flexible Replication Responsiveness (*Axiom FRR*): A poverty measure $P: \mathbf{X} \times \mathrm{R}_{++} \rightarrow \mathrm{R}$ satisfies Flexible Replication Responsiveness if and only if, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and $\mathrm{z} \in \mathrm{R}_{++}$, if $\mathbf{y} = (\mathbf{x}, ..., \mathbf{x})$ and $n(\mathbf{y}) = kn(\mathbf{x})$ for any $k \in \mathrm{N}$, then $P(\mathbf{x}; \mathrm{z}) = k^{\beta} P(\mathbf{y}; \mathrm{z})$, where $0 < \beta < 1$.

Notice now that as we move from 0 to 1, we move from endorsement of the Replication Invariance Axiom toward endorsement of the Replication Scaling Axiom. It is reasonable to suggest that a properly centrist view of poverty, which lies mid-way between Invariance and Scaling, is yielded by a value for β of one-half. A natural question which arises in this context is: how may we operationalize a ‘centrist’ headcount measure of poverty, one which strikes a balance between the demands of the headcount ratio and the aggregate headcount?

Without getting into the details of work done elsewhere, Subramanian (2005a) provides a simple axiomatic justification for a ‘flexible headcount measure’ of poverty (P_{β}) which is given, for all $(\mathbf{x}; z) \in \mathbf{X} \times \mathrm{R}_{++}$, by:

(4.3) $P_{\beta}(\mathbf{x}; z) = P_A^{\beta}(1 + P_H)$.

When β is one-half, we obtain a 'properly centrist' poverty measure (P_C), given, for all $(\mathbf{x};z) \in \mathbf{X} \times \mathrm{R}_{++}$, by:

(4.4) $P_C(\mathbf{x};\mathrm{z}) = \sqrt{P_A}(1+P_H)$.

Equation (4.4) is a slightly more elaborate product rule than one proposed by Arriaga (1970) in the context of the measurement of urbanization. While the *proportion* of a country's population living in its cities has conventionally been taken to be an appropriate headcount indicator of urbanization, Arriaga also advanced the rival claim of the *absolute size* of the country's population living in its cities. England has larger cities than Hawaii (contrast London and Honolulu), but the import of this fact could be diluted if we went only by the proportion of the city-inhabited populations in the two countries. Arriaga's suggested measure of population is one which takes account of both the headcount ratio and the aggregate headcount, combined multiplicatively. In the context of poverty measurement, and in terms of the notation we have employed, Arriaga's measure (translated from urbanization to poverty) - call it $P_{ARRIAGA}$ - is given, for all $(\mathbf{x};z) \in \mathbf{X} \times \mathrm{R}_{++}$, by:

(4.5) $P_{ARRIAGA}(\mathbf{x};z) = P_H(\mathbf{x};z)P_A(\mathbf{x};z)\ [= n_Q^2(\mathbf{x};z)/n(\mathbf{x};z)]$.

$P_{ARRIAGA}$ can be interpreted as 'the expected value of poverty' in a society. To see this, consider a dichotomous indicator of poverty status, given by

$d_i = 1$ if person *i* is poor;

= 0 if person *i* is non-poor.

In a population of *n* persons (constituting the set *N*) of whom n_Q are poor, the probability that any person *i* chosen at random is poor is given by $p_i = n_Q/n$, and the expected value of poverty in the community (which is the probability-weighted sum of each person's poverty status) is given by: $\mathrm{E} = \sum_{i=1}^{n} p_i d_i = (n_Q/n)(\sum_{i \in Q(N)} 1) + (1-n_Q/n)(\sum_{i \in R(N)} 0) = n_Q^2/n \equiv P_{ARRIAGA}$.
The Arriaga measure reflects the simplest of product rules: it is the product of a relative headcount measure of poverty (the headcount ratio) and an absolute measure (the aggregate headcount): this is exactly analogous to the Krtscha intermediate measure of inequality which, to recall, is the product of a relative measure of inequality (the coefficient of variation) and an absolute measure (the standard deviation). As noted earlier, and as can be seen from (4.4) and (4.5), the measure P_C is a slightly elaborate version of the measure $P_{ARRIAGA}$: one noteworthy difference between the two is that $Lim_{P_H \to 0} P_{ARRIAGA} = 0$, whereas - and perhaps more reasonably - $Lim_{P_H \to 0} P_C = \sqrt{P_A}$. In general, the product rule allows us

to realize a useful outcome: when two distributions are indistinguishable from each other in terms of the headcount ratio, the decisive ranking is performed by the aggregate headcount; and when two distributions are indistinguishable from each other in terms of the aggregate headcount, the decisive ranking is performed by the headcount ratio. This, as observed by Chakravarty, Kanbur and Mukherjee (2006), is a nice property in a headcount measure which avoids the extreme values of relativity and absoluteness encompassed in the headcount ratio and the aggregate headcount respectively.

It may be noted, finally, that Replication Invariance requires (within its domain of application) the same sort of value-neutrality which Scale Invariance does (within *its* domain of application). Such cardinal invariance is sensibly replaced by a less demandingly ordinal, *ranking*-related invariance. A requirement of this nature need demand only that if $\mathbf{x}'$ is a *k*-replication of $\mathbf{x}$ and $\mathbf{y}'$ is a *k*-replication of $\mathbf{y}$, with $\mathbf{x}$ and $\mathbf{y}$ being of the same dimensionality, then $[I(\mathbf{x}) \geq I(\mathbf{y})]$ implies that $[I(\mathbf{x}') \geq I(\mathbf{y}')]$. This is encompassed in what may be called the *Population Consistency Axiom* (*Axiom PC*) [which Zoli (2009) calls the Population Replication Principle, and which has been discussed in Dalton (1920), Bossert (1990b) and Zoli (2009)]. Axiom PC stands in the same relation to Axiom RI as Zheng's Axiom UC stands in relation to Axiom SI. It is useful to note that the poverty measures P_A, P_C and $P_{ARRIAGA}$ all satisfy the Population Consistency Axiom.

5. SUMMARY AND CONCLUSIONS

The present paper has been an essentially expository and clarificatory essay dealing with the merits of two very widely-employed axioms in the measurement of inequality and poverty. These are, respectively, the Scale Invariance Axiom and the Replication Invariance Axiom. The former proposes that inequality is best assessed on a *per rupee* basis, and the latter that poverty is best assessed on a *per person* basis. In their respective ways, the two axioms have championed the cause of a fundamentally *relative* approach to the assessment and measurement of inequality and poverty.

While these relative approaches continue to enjoy an overwhelmingly popular vogue in the measurement literature, they have also been challenged (albeit in a somewhat small minority voice) by some practitioners. This paper has briefly reviewed some of the *absolute* approaches to measurement that have been advanced in both the inequality and the poverty literature. The paper has also examined some of the logical and ethical difficulties inherent in the extreme orientations of any purely relative or purely absolute approach to measurement. This has led to an evaluation of *intermediate* measures that have been suggested in the literature. It has also been pointed out that Krtscha's intermediate measure of inequality exploits a simple product rule to express the inequality measure as a function of a relative measure (the coefficient of variation) and an absolute measure (the standard deviation). Precisely analogously, a poverty measure which is a simple translation of an urbanization index due to Arriaga is shown to be a product of the (relative) headcount ratio and the (absolute) aggregate headcount.

It is no doubt true that these intermediate measures reflect some of the vices of their respective relative and absolute ancestors, but the combination of these mutually antagonistic sets of vices does tend to mitigate their respective 'stand-alone' excesses. This paper is an invitation to acknowledge the desirability of incorporating intermediate approaches to the measurement of inequality and poverty more routinely into mainstream treatments of the relevant theoretical and applied literature. Failing this, Scale Invariance and Replication Invariance must continue to remain two prominent elephants in the living room of Distributional Analysis.

REFERENCES

Arriaga, E.E. (1970). `A New Approach to the Measurement of Urbanization', *Economic Development and Cultural Change*, 18(2): 206-218.

Atkinson, A. B. and A. Brandolini (2004). 'Global World Inequality: Absolute, Relative or Intermediate?', paper prepared for the 28th General Conference of the International Association for Research in Income and Wealth, Cork, Ireland, august 22–28, 2004. Available at: http://www.iariw.org/papers/2004/brand.pdf .

Azpitarte, F. and O. Alsonso-Villar (2011). 'Ray-Invariant Intermediate Inequality Measures: A Lorenz Dominance Criterion', ECINEQ Working paper 2011-26.

Bosmans, K., K. Decancq and A. Decoster (2011). 'The Evolution of Global Inequality: Absolute, Relative and Intermediate Views', Center for Economic Studies Discussion Paper Series (DPS) 11.03, Katholieke Universiteit Leuven.

Bossert, W. (1990a). 'Social Evaluation with Variable Population Size: An Alternative Concept', *Mathematical Social Sciences*, 19(2): 143-158.

Bossert, W. (1990b). 'Population Replication and Ethical Poverty Measurement', *Mathematical Social Sciences*, 20(3): 227-238.

Bossert, W. and A. Pfingsten (1990). 'Intermediate inequality: Concepts, Indices and Welfare Implications', *Mathematical Social Sciences*, 19(2): 117-134.

Broome, J. (1996). `The Welfare Economics of Population', *Oxford Economic Papers*, 48(2): 177-193.

Chakravarty, S., S. R. Kanbur and D. Mukherjee (2006). 'Population Growth and Poverty Measurement', *Social Choice and Welfare* 26(3): 471-483.

Chakravarty, S. R. and S. Tyagarupananda (1998). 'The Sugroup Decomposable Absolute Indices of Inequality', in S. R. Chakravarty, D. Coondoo and R. Mukherjee (eds.): *Quantitative Economics: Theory and Practice, Essays in Honour of Professor N. Bhattacharya*. New Delhi: Allied Publishers Limited.

Chakravarty, S. R. and S. Tyagarupananda (2009). 'The Subgroup Decomposable Intermediate Indices of Inequality', *Spanish Economic Review*, 11(2), 83-97.

Dalton, H. (1920). 'The Measurement of the Inequality of Incomes', *Economic Journal*, 30(119): 348-361.

Del Rio, C. and O. Alonso-Villar (2008). 'New Unit-Consistent Intermediate Inequality Indices', *Economic Theory*, 42 (3), 505-521.

Del Rio, C. and O. Alonso-Villar (2011). 'Rankings of Income Distributions: A Note on Intermediate Inequality Indices', Universidade de Vigo. Available at: http://webs.urigo.es/crio/REI.pdf .

Del Rio, C. and J. Ruiz-Castillo (2000). 'Intermediate Inequality and Welfare', *Social Choice and Welfare*, 17 (2), 223-239.

Del Rio, C. and J. Ruiz-Castillo (2001). 'Intermediate Inequality and Welfare: The Case of Spain', *Review of Income and Wealth*, 47 (2), 221-237.

Hassoun, N. (2010). 'Another Mere Addition Paradox? A Problem for Some Common Poverty Indexes in Variable Populations', WIDER Working Paper 2010/120, URL: http://www.wider.unu.edu/publications/working-papers/2010/en_GB/wp2010-120/.

Hassoun, N. and S. Subramanian (2012). 'An Aspect of Variable Population Poverty Comparisons', *Journal of Development Economics*, 98(2): 238-241.

Jenkins, S. P. and M. Jäntti (2005). 'Methods for Summarizing and Comparing Wealth Distributions', ISER Working Paper 2005-05. Colchester: University of Essex, Institute for Social and Economic Research.

Kanbur, S. R and D. Mukherjee (2007). 'Premature Mortality and Poverty Measurement', *Bulletin of Economic Research*, 19(4): 339-359.

Kolm, S. Ch. (1976a). 'Unequal Inequalities I', *Journal of Economic Theory*, 12(3), 416-454.

Kolm, S. Ch. (1976b). 'Unequal Inequalities II', *Journal of Economic Theory*, 13(1), 82-111.

Krtscha, M. (1994). 'A New Compromise Measure of Inequality', in W. Eichhorn (ed.): *Models and Measurement of Welfare and Inequality*. Heidelberg: Springer-Verlag.

Kundu, A. and T. E. Smith (1983). 'An Impossibility Theorem on Poverty Indices', I*nternational Economic Review* 24(2): 423-34.

Moyes, P. (1987). 'A New Concept of Lorenz Domination', *Economics Letters*, 23 (2), 203-207.

Paxton, J. (2003). 'A Poverty Outreach Index and its Application to Microfinance', *Economics Bulletin* 9(2): 1-10.

Sen, A.K. (1976). `Poverty: An Ordinal Approach to Measurement', *Econometrica*, 44(2): 219-31.

Sen, A.K. (1981). *Poverty and Famines: An Essay on Entitlement and Deprivation*. Oxford: Clarendon.

Subramanian, S. (2000). 'Poverty Measurement and the Repugnant Conclusion', S. Guhan Memorial Series Discussion Paper 3, Madras Institute of Development Studies, Chennai.

Subramanian, S. (2002). `Counting the Poor: An Elementary Difficulty in the Measurement of Poverty', *Economics and Philosophy*, 277-285.

Subramanian, S. (2005a). `Fractions *versus* Whole Numbers: On Headcount Comparisons of Poverty Across Variable Populations', *Economic and Political Weekly*, 40(43): 4625-4628.

Subramanian, S. (2005b). 'Headcount Poverty Comparisons', International Poverty Centre *One Pager* No. 18, United Nations Development Programme: Brasilia, Brazil.

Subramanian, S. (2006). 'Poverty Measurement and Theories of Beneficence', in S. Subramanian: *Rights, Deprivation, and Disparity: Essays in Concepts and Measurement* (Collected Essays Series). Oxford University Press: Delhi. 2006.

Subramanian, S. (20012). 'The Focus Axiom and Poverty: On the Co-existence of Precise Language and Ambiguous Meaning in Economic Mesurement', *Economics: The Open-Access, Open-Assessment E-Journal*, Vol. 6, 2012-8. http://dx.doi.org/10.5018/economics-ejournal.ja.2012-8.

Subramanian, S. (2013a). 'Assessing Inequality in the Presence of Growth: An Expository Essay', paper presented at a conference on 'Inclusive Growth in Africa: Measurement, Causes, and Consequences', organized by UNU-WIDER and held over September 20th and 21st, 2013, at Helsinki, Finland. Available at: http://www1.wider.unu.edu/inclusivegrowth/sites/default/files/IGA/Subramanian.pdf .

Subramanian, S. (2013b). 'Dividing a Cake (or) Distributional Values in the Measurement of Economic Inequality: An Expository Note', forthcoming in *Journal of Philosophical Economics*.

Yoshida, T. (2005). 'Social Welfare Rankings of Income Distributions: A New Parametric Concept of Intermediate Inequality', *Social Choice and Welfare*, 24 (3): 557-574.

Zheng, B. (2007). 'Unit-Consistent Decomposable Inequality Measures', *Economica*, 74 (293): 97-111.

Zoli, C. (2009). 'Variable Population Welfare and Poverty Orderings Satisfying Replication Properties', Department of Economics (University of Verona) Working Paper 69. Available at: http://dse.univr.it/workingpapers/WP6909.pdf .

Zoli, C. (2012). 'Characterizing Inequality Equivalence Criteria', Department of Economics (University of Verona) Working Paper 32. Available at:

http://leonardo3.dse.univr.it/home/workingpapers/IneqEquivRevision20wp20201220VR.pdf